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RENORMALIZATION OF GAUGE THEORIES AND MASTER EQUATION

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ABSTRACT

The evolution of ideas which has led from the first proofs of the renormalizability of non-abelian gauge theories, based on Slavnov–Taylor identities, to the modern proof based on the BRS symmetry and the *master equation* is recalled. This lecture has been delivered at the **Symposium in the Honour of Professor C. N. Yang**, Stony-Brook, May 21-22 1999.

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1 Introduction

It is a rare privilege for me to open this conference in honour of Professor Yang. His scientific contributions have been for me an essential source of inspiration. The most obvious example, Yang–Mills fields or gauge theories, will be illustrated by my talk. But there are other important aspects of Pr. Yang’s work which have also directly influenced me: Professor Yang has consistently shown us that a theorist could contribute to quite different domains of physics like Particle Physics, the Statistical Physics of phase transitions or integrable systems.... Moreover his work has always emphasized mathematical elegance.

Finally by offering me a position at the ITP in Stony-Brook in 1971, Pr. Yang has given me the opportunity to start with the late Benjamin W. Lee a work on the renormalization of gauge theories, which has kept me busy for several years and played a major role in my scientific career.

Let me add a few other personal words. The academic year 1971–1972 I spent here at the ITP has been of the most exciting and memorable of my scientific life. One reason of course is my successful collaboration with Ben Lee. However another reason is the specially stimulating atmosphere Professor Yang had managed to create at the ITP, by attracting talented physicists, both ITP members and visitors, by the style of scientific discussions, seminars and lectures.

My interest in Yang–Mills fields actually dates back to 1969, and in 1970 I started a work, very much in the spirit of the original paper of Yang and Mills, on the application of massive Yang–Mills fields to Strong Interaction dynamics. Although in our work massive Yang–Mills fields were treated in the spirit of effective field theories, we were aware of the fact that such a quantum field theory was not renormalizable.

In the summer of 1970 I presented the preliminary results of our work in a summer school in Cargèse, where Ben Lee was lecturing on the renormalization of spontaneous and linear symmetry breaking. This had the consequence that one year later I arrived here at the ITP to work with him.

Ben had just learned, in a conference I believe, from ’t Hooft’s latest work on the renormalizability of non-abelian gauge theories both in the symmetric and spontaneously broken phase and was busy proving renormalizability of the abelian Higgs model. We immediately started our work on the much more involved non-abelian extension.

Our work was based on functional integrals and functional methods and a generalization of so-called Slavnov–Taylor identities, consequence of the properties of the Faddeev–Popov (FP) determinant arising in the quantization of gauge theories. In a series of four papers (1972–1973), we examined most aspects of the renormalization of gauge theories.

2 Classical gauge action and quantization

The principle of gauge invariance which promotes continuous global (or rigid) symmetries to local (gauge) symmetries provides a beautiful geometrical method to generate interactions between particles. The pure Yang–Mills action has the form

$$\mathcal{S}(\mathbf{A}_\mu) = -\frac{1}{4e^2} \int d^d x \operatorname{tr} \mathbf{F}_{\mu\nu}^2(x),$$

where $\mathbf{A}_\mu(x)$ is the gauge field, a matrix belonging to the Lie algebra of the symmetry group, and $\mathbf{F}_{\mu\nu}(x)$ the associated curvature obtained from the covariant derivative \mathbf{D}_μ

$$\mathbf{D}_\mu = \partial_\mu + \mathbf{A}_\mu,$$

by

$$\mathbf{F}_{\mu\nu}(x) = [\mathbf{D}_\mu, \mathbf{D}_\nu] = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu].$$

Matter fields which transform non-trivially under the group will then be coupled to the gauge field. For fermions the action takes the typical form

$$\mathcal{S}_F(\bar{\psi}, \psi) = - \int d^d x \bar{\psi}(x) (\not{D} + M) \psi(x),$$

and for the boson fields:

$$\mathcal{S}_B(\phi) = \int d^d x \left[(\mathbf{D}_\mu \phi)^\dagger \mathbf{D}_\mu \phi + V(\phi) \right],$$

in which $V(\phi)$ is a group invariant function of the scalar field ϕ .

Quantization. The classical action results from a beautiful construction, but the quantization apparently completely destroys the geometric structure. Due to the gauge invariance the degrees of freedom associated with gauge transformations have no dynamics and therefore a straightforward quantization of the classical action does not generate a meaningful perturbation theory (though non-perturbative calculations in lattice regularized gauge theories can be performed). It is thus necessary to *fix* the gauge, a way of expressing that some dynamics has to be provided for these degrees of freedom. For example, motivated by Quantum Electrodynamics, one may add to the action a covariant non-gauge invariant contribution

$$\mathcal{S}_{\text{gauge}} = \frac{1}{2\xi e^2} \int d^d x \operatorname{tr} (\partial_\mu \mathbf{A}_\mu)^2. \quad (2.1)$$

However, simultaneously, and this is a specificity of non-abelian gauge theories, it is necessary to modify the functional integration measure of the gauge field to maintain formal unitarity. In the case of Landau's gauge (2.1) one finds

$$[d\mathbf{A}_\mu(x)] \mapsto [d\mathbf{A}_\mu(x)] \det \mathbf{M}, \quad (2.2)$$

where \mathbf{M} is the operator

$$\mathbf{M}(x, y) = \partial_\mu \mathbf{D}_\mu \delta(x - y).$$

This (Faddeev–Popov) determinant is the source of many difficulties. Indeed after quantizing the theory one has to renormalize it. Renormalization is a theory of deformations of local actions. However the determinant generates a non-local contribution to the action. Of course, using a well-known trick, it is possible to rewrite the determinant as resulting from the integration over unphysical spin-less fermions $\mathbf{C}, \bar{\mathbf{C}}$ (the “ghosts”) of an additional contribution to the action

$$\mathcal{S}_{\text{ghosts}} = \int d^d x \bar{\mathbf{C}}(x) \partial_\mu \mathbf{D}_\mu \mathbf{C}(x).$$

After this transformation the action is local and renormalizable in the sense of power counting. However, in this local form all traces of the original symmetry seem to have been lost.

3 Renormalization

The measure (2.2) is the invariant measure for a set of non-local transformations which for infinitesimal transformations takes the form

$$\delta \mathbf{A}_\mu(x) = \int dy \mathbf{D}_\mu \mathbf{M}^{-1}(x, y) \omega(y),$$

the field $\omega(x)$ parametrizing the transformation. Using this property it is possible to derive a set of Ward–Takahashi (Slavnov–Taylor) identities between Green’s functions and to prove renormalizability of gauge theories both in the symmetric and spontaneously broken Higgs phase. The non-local character of these transformations and the necessity of using two different representations, one non-local but with invariance properties, the other one local and thus suitable for power counting analysis, explains the complexity of the initial proofs.

Though the problem of renormalizing gauge theories could then be considered as settled, one of the remaining problems was that the proofs, even in the most synthetic presentation like in Lee–Zinn-Justin IV, were complicated, non-transparent, and more based on trial and error than systematic methods.

Returning to Saclay I tried to systematize the renormalization program of quantum field theories with symmetries. I abandoned the determination of renormalization constants by relation between Green’s functions, for a more systematic approach based on loop expansion and counter-terms.

The idea is to proceed by induction on the number of loops. Quickly summarized:

One starts from a regularized local lagrangian with some symmetry properties. One derives, as consequence of the symmetry, identities (generally called Ward–Takahashi or WT identities) satisfied by the generating functional Γ of one-particle irreducible (1PI) Green’s functions (or proper vertices). By letting the

cut-off go to infinity (or the dimension to four in dimensional regularization) one obtains identities satisfied by the sum Γ_{div} of all divergent contributions at one loop order. At this order Γ_{div} is a local functional of a degree determined by power counting. By subtracting Γ_{div} from the action one obtains a theory finite at one-loop order. One then reads off the symmetry of the lagrangian renormalized at one-loop order and repeats the procedure to renormalize at two-loop order. The renormalization program is then based on determining general identities valid both for the action and the 1PI functional, which are stable under renormalization, i.e. stable under all deformations allowed by power counting. One finally proves the stability by induction on the number of loops.

Unfortunately this program did not apply to non-abelian gauge theories, because it required a symmetry of the local quantized action, and none was apparent. WT identities were established using symmetry properties of the theory in the non-local representation

In the spring of 1974 my student Zuber drew my attention to a preliminary report of a work of Becchi, Rouet and Stora who had discovered a strange fermion-type (like supersymmetry) symmetry of the complete quantized action including the ghost contributions. There were indications that this symmetry could be used to somewhat simplify the algebra of the proof of renormalization. Some time later, facing the daunting prospect of lecturing about renormalization of gauge theories and explaining the proofs to non-experts, I decided to study the BRS symmetry. I then realized that the BRS symmetry was the key allowing the application of the general renormalization scheme and in a summer school in Bonn (1974) I presented a general proof of renormalizability of gauge theories based on BRS symmetry and the *master equation*.

4 BRS symmetry

The form of the BRS transformations in the case of non-abelian gauge transformations is rather involved and hides its simple origin. We thus give here a presentation which shows how BRS symmetry arises in apparently a simpler context. Let φ^α be a set of dynamical variables satisfying a system of equations:

$$E_\alpha(\varphi) = 0, \quad (4.1)$$

where the functions $E_\alpha(\varphi)$ are smooth, and $E_\alpha = E_\alpha(\varphi)$ is a one-to-one map in some neighbourhood of $E_\alpha = 0$ which can be inverted in $\varphi^\alpha = \varphi^\alpha(E)$. This implies in particular that the equation (4.1) has a unique solution φ_s^α . We then consider some function $F(\varphi)$ and we look for a formal representation of $F(\varphi_s)$, which does not require solving equation (4.1) explicitly. We can then write:

$$\begin{aligned} F(\varphi_s) &= \int \left\{ \prod_\alpha dE^\alpha \delta(E_\alpha) \right\} F(\varphi(E)) \\ &= \int \left\{ \prod_\alpha d\varphi^\alpha \delta[E_\alpha(\varphi)] \right\} \mathcal{J}(\varphi) F(\varphi), \end{aligned} \quad (4.2)$$

with:

$$\mathcal{J}(\varphi) = \det \mathbf{E}, \quad E_{\alpha\beta} \equiv \frac{\partial E_\alpha}{\partial \varphi^\beta}.$$

We have chosen $E_\alpha(\varphi)$ such that $\det \mathbf{E}$ is positive.

Slavnov–Taylor identity. The measure $d\rho(\varphi)$:

$$d\rho(\varphi) = \mathcal{J}(\varphi) \prod_{\alpha} d\varphi^\alpha, \quad (4.3)$$

has a simple property. The measure $\prod_{\alpha} dE_\alpha$ is the invariant measure for the group of translations $E_\alpha \mapsto E_\alpha + \nu_\alpha$. It follows that $d\rho(\varphi)$ is the invariant measure for the translation group non-linearly realized on the new coordinates φ_α (provided ν_α is small enough):

$$\varphi^\alpha \mapsto \varphi'^\alpha \quad \text{with} \quad E_\alpha(\varphi') - \nu_\alpha = E_\alpha(\varphi). \quad (4.4)$$

This is the origin, in gauge theories, of the Slavnov–Taylor symmetry.

The infinitesimal form of the transformation law can be written more explicitly:

$$\delta\varphi^\alpha = [E^{-1}(\varphi)]^{\alpha\beta} \nu_\beta. \quad (4.5)$$

BRS symmetry. Let us again start from identity (4.2) and first replace the δ -function by its Fourier representation:

$$\prod_{\alpha} \delta[E_\alpha(\varphi)] = \int \prod_{\alpha} \frac{d\lambda^\alpha}{2i\pi} e^{-\lambda^\alpha E_\alpha(\varphi)}. \quad (4.6)$$

The λ -integration runs along the imaginary axis. From the rules of fermion integration we know that we can also write the determinant as an integral over Grassmann variables c^α and \bar{c}^α :

$$\det \mathbf{E} = \int \prod_{\alpha} (dc^\alpha d\bar{c}^\alpha) \exp(\bar{c}^\alpha E_{\alpha\beta} c^\beta). \quad (4.7)$$

Expression (4.2) then takes the apparently more complicated form

$$F(\varphi_s) = \mathcal{N} \int \prod_{\alpha} (d\varphi^\alpha dc^\alpha d\bar{c}^\alpha d\lambda^\alpha) F(\varphi) \exp[-S(\varphi, c, \bar{c}, \lambda)], \quad (4.8)$$

in which \mathcal{N} is a constant normalization factor and $S(\varphi, c, \bar{c}, \lambda)$ the quantity:

$$S(\varphi, c, \bar{c}, \lambda) = \lambda^\alpha E_\alpha(\varphi) - \bar{c}^\alpha E_{\alpha\beta}(\varphi) c^\beta. \quad (4.9)$$

While we seem to have replaced a simple problem by a more complicated one, in fact in many situations (and this includes the case where equation (4.1) is a field equation) it is easy to work with the integral representation (4.8).

Quite surprisingly the function S has a symmetry, which actually is a consequence of the invariance of the measure (4.3) under the group of transformations (4.5). This BRS symmetry, first discovered in the quantization of gauge theories by Becchi, Rouet and Stora (BRS), is a fermionic symmetry in the sense that it transforms commuting variables into Grassmann variables and vice versa. The parameter of the transformation is a Grassmann variable, an anti-commuting constant $\bar{\varepsilon}$. The variations of the various dynamic variables are:

$$\begin{cases} \delta\varphi^\alpha = \bar{\varepsilon}c^\alpha, & \delta c^\alpha = 0, \\ \delta\bar{c}^\alpha = \bar{\varepsilon}\lambda^\alpha, & \delta\lambda^\alpha = 0, \end{cases} \quad (4.10)$$

with:

$$\bar{\varepsilon}^2 = 0, \quad \bar{\varepsilon}c^\alpha + c^\alpha\bar{\varepsilon} = 0, \quad \bar{\varepsilon}\bar{c}^\alpha + \bar{c}^\alpha\bar{\varepsilon} = 0.$$

The transformation is obviously *nilpotent* of vanishing square: $\delta^2 = 0$.

The BRS transformation can be represented by a Grassmann differential operator \mathcal{D} , when acting on functions of $\{\varphi, c, \bar{c}, \lambda\}$:

$$\mathcal{D} = c^\alpha \frac{\partial}{\partial\varphi^\alpha} + \lambda^\alpha \frac{\partial}{\partial\bar{c}^\alpha}. \quad (4.11)$$

The nil-potency of the BRS transformation is then expressed by the identity:

$$\mathcal{D}^2 = 0. \quad (4.12)$$

5 The master equation

In gauge theories the role of the φ variables is played by the group elements which parametrize gauge transformations and the equation (4.1) is simply the gauge fixing equation. The form of the BRS transformation is more complicated only because it is written in terms of group elements:

$$\begin{cases} \delta\mathbf{A}_\mu(x) = -\bar{\varepsilon}\mathbf{D}_\mu\mathbf{C}(x), & \delta\mathbf{C}(x) = \bar{\varepsilon}\mathbf{C}^2(x), \\ \delta\bar{\mathbf{C}}(x) = \bar{\varepsilon}\lambda(x), & \delta\lambda(x) = 0. \end{cases} \quad (5.1)$$

However this form of BRS transformations is not stable under renormalization because the form of the gauge transformations is modified by the renormalization.

To discuss renormalization it is necessary to add to the action two sources \mathbf{K}_μ , \mathbf{L} , for the BRS transformations which are not linear in the fields

$$\mathcal{S} \mapsto \mathcal{S} + \int d^4x \operatorname{tr} \left(-\mathbf{K}_\mu(x)\mathbf{D}_\mu\mathbf{C}(x) + \mathbf{L}(x)\mathbf{C}^2(x) \right).$$

The sources for BRS transformations, \mathbf{K}_μ and \mathbf{L} , have been later renamed anti-fields.

The stable relation satisfied by the complete action, including these additional contributions, then takes a, at first sight disappointingly simple, quadratic form (here written in a simple example, without matter fields)

$$\int d^4x \left(\frac{\delta \mathcal{S}}{\delta A_\mu^\alpha(x)} \frac{\delta \mathcal{S}}{\delta K_\mu^\alpha(x)} + \frac{\delta \mathcal{S}}{\delta C^\alpha(x)} \frac{\delta \mathcal{S}}{\delta L^\alpha(x)} + \lambda^\alpha(x) \frac{\delta \mathcal{S}}{\delta \bar{C}^\alpha(x)} \right) = 0. \quad (5.2)$$

In particular the master equation (5.2) contains no explicit reference to the initial gauge transformation. Therefore one may worry that it does not determine the renormalized action completely, and that the general renormalization program fails in the case of non-abelian gauge theories. However, one slowly discovers that the master equation has remarkable properties. In particular all its local solutions which satisfy the power counting requirements, have indeed the form of an action for a quantized non-abelian gauge theory. Then continuity implies, in the semi-simple example at least, preservation of all geometric properties.

One surprising outcome still bothered me for some time: The master equation has solutions with quartic ghost interactions, which cannot be obviously related to a determinant. On the other hand the master equation by itself (and this one of its main properties) implies gauge independence and unitarity.

Only a few years later, elaborating on a remark of Slavnov, was I able to reproduce a general quartic ghost term as resulting from a generalized gauge fixing procedure (Zinn-Justin 1984).

After the renormalization program was successfully completed, one important problem remained, of relevance for instance to the description of deep-inelastic scattering experiments: the renormalization of gauge invariant operators of dimension higher than four. Using similar techniques Stern-Kluberg and Zuber were able to solve the problem for operators of dimension six and conjecture the general form. Only recently has the general conjecture been proven rigourously by non-trivial cohomology techniques (Barnich, Brandt and Henneaux 1995).

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Bibliographical Notes

After the fundamental article

- C.N. Yang and R.L. Mills, *Phys. Rev.* 96 (1954) 191,
the main issue was the quantization of gauge theories
R.P. Feynman, *Acta Phys. Polon.* 24 (1963) 697;
B.S. DeWitt, *Phys. Rev.* 162 (1967) 1195, 1239;

L.D. Faddeev and V.N. Popov, *Phys. Lett.* 25B (1967) 29;

S. Mandelstam, *Phys. Rev.* 175 (1968) 1580.

In the following article we tried to apply the idea of massive Yang–Mills fields to Strong Interaction Dynamics

J.L. Basdevant and J. Zinn-Justin, *Phys. Rev.* D3 (1971) 1865.

Among the articles discussing Ward–Takahashi and renormalization see for instance

G. 't Hooft, *Nucl. Phys.* B33 (1971) 173; *ibid.* B35 (1971) 167.

A.A. Slavnov, *Theor. Math. Phys.* 10 (1972) 99.

J.C. Taylor, *Nucl. Phys.* B33 (1971) 436.

B.W. Lee and J. Zinn-Justin, *Phys. Rev.* D5 (1972) 3121, *ibid.* 3137, *ibid.* 3155; *ibid.* D7 (1973) 1049.

G. 't Hooft and M. Veltman, *Nucl. Phys.* B50 (1972) 318.

D.A. Ross and J.C. Taylor, *Nucl. Phys.* B51 (1973) 125;

B.W. Lee, *Phys. Lett.* 46B (1973) 214; *Phys. Rev.* D9 (1974) 933.

The anti-commuting type symmetry of the quantized action is exhibited in

C. Becchi, A. Rouet and R. Stora, *Comm. Math. Phys.* 42 (1975) 127.

Most of the preceding articles are reprinted in

Selected papers on *Gauge Theory of Weak and Electromagnetic Interactions*, C.H. Lai ed., World Scientific (Singapore 1981).

The general proof, based on BRS symmetry and the master equation, of renormalizability in an arbitrary gauge, can be found in the proceedings of the Bonn summer school 1974,

J. Zinn-Justin in *Trends in Elementary Particle Physics (Lectures Notes in Physics 37)*, H. Rollnik and K. Dietz eds. (Springer-Verlag, Berlin 1975).

See also

J. Zinn-Justin in *Proc. of the 12th School of Theoretical Physics, Karpacz 1975*, Acta Universitatis Wratislaviensis 368;

B.W. Lee in *Methods in Field Theory*, Les Houches 1975, R. Balian and J. Zinn-Justin eds. (North-Holland, Amsterdam 1976).

Finally a systematic presentation can be found in

J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon Press (Oxford 1989, third ed. 1996).

For an alternative proof based on BRS symmetry and the BPHZ formalism see

C. Becchi, A. Rouet and R. Stora, *Ann. Phys. (NY)* 98 (1976) 287.

Non-linear gauges and the origin of quartic ghost terms are investigated in

J. Zinn-Justin, *Nucl. Phys.* B246 (1984) 246.

Renormalization of gauge invariant operators and the BRST cohomology are discussed in

H. Kluberg-Stern and J.-B. Zuber, *Phys. Rev.* D12 (1975) 467;

G. Barnich, and M. Henneaux, *Phys. Rev. Lett.* 72 (1994) 1588; G. Barnich, F. Brandt and M. Henneaux, *Comm. Math. Phys.* 174 (1995) 93.